The Eigenfunction Expansion of Dyadic Green's Functions for Chirowaveguides

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***Abstract*— A general method of formulating eigenfunction expansion of dyadic Green's functions in lossless, reciprocal and homogeneous chirowaveguides is presented. Bohren's decomposition of the electromagnetic field is used to obtain the vector wave functions. The method of $\bar{\bar{G}}_m$ is used to rigorously derive the magnetic and electric dyadic Green's functions. A specific application to the cylindrical chirowaveguide illustrates the method.**

I. Introduction

RECENTLY, the theory of chirowaveguide has been a topic of hot research. The propagation characteristics of electromagnetic waves in chirowaveguides have been deeply investigated by many authors [1]–[11]. The increasing interest in such devices shows their potential applications in the area of electromagnetics. Basically the presence of chirality in a medium rotates the plane of polarization of an electromagnetic wave. Chirality means the rotations in two different directions are different so that a handedness of the medium is manifested. This phenomenon has been discovered very early in chemistry. Its interest in electromagnetics was first noted in the field of optics. (A quite detailed description of the historical background of chirality can be found in the work of S. Bassiri [12].) The application of chirality to microwaves and millimeter waves is only a recent matter due to the possibility of fabricating chiral materials for such frequency ranges [13].

Dyadic Green's functions relate a current source to its fields. Hence they are important in the excitation aspect of waveguides such as the determination of the feed point impedance. Dyadic Green's functions in an unbounded chiral medium [14], [15] as well as in the presence of a chiral sphere [16] have been formulated. One- and two-dimensional (2-D) dyadic Green's functions in chiral media have also been obtained [17]. Although Engheta *et al.* [16] sought an eigenfunction expansion of the electric dyadic Green's function with the spherical vector wave functions for the case of scattering from a chiral sphere, their result is not a complete expansion [18] and only applicable to source free regions. More recently, in Li's [19] formulation of the dyadic Green's functions for a radially multilayered chiral sphere, the singular term accounting for the electric field in the source point was reinstated but the reason was not explained.

In this paper, we provide a method of rigorous formulations of the eigenfunction expansion of dyadic Green's functions in lossless, reciprocal and homogeneous chirowaveguides. We use the method of $\bar{\bar{G}}_m$ [20]. Not only is the electric dyadic Green's functions obtained, but also the magnetic dyadic Green's functions. The singular term in the electric dyadic Green's function is shown to be a natural outcome of the $\bar{\bar{G}}_m$ method. A specific application of the method to the cylindrical chirowaveguide is given to demonstrate the detailed formulation steps. We hope this work will be useful and illustrative.

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II. Formulation

A. Constitutive Equations

In a source-free region with a chiral medium, the constitutive equations have been proposed by several authors. A detailed description of the different forms of the constitutive equations and the conditions on their mutual equivalence have been given by Lakhtakia *et al.* [15]. Among the various forms of the constitutive equations, the one deduced by Post [21], i.e.,

$$\vec{D} = \varepsilon\vec{E} + j\gamma\vec{B} \tag{1a}$$
$$\vec{H} = j\gamma\vec{E} + (1/\mu)\vec{B} \tag{1b}$$

is to be used in our present study. This is because it has a simple expression and is also supported by experimental studies [21], [22]. In (1a) and (1b), ε, μ and γ represent, respectively, permittivity, permeability and chirality admittance of a lossless and reciprocal chiral medium. The fact that the divergences of different sides of (1b) are not equal means that it is only applicable to a source free region. When a source is present, (1b) must be modified as follows:

$$\vec{H} = j\gamma\vec{E} + (1/\mu)\vec{B} - [\gamma/(\omega\varepsilon)]\vec{J} \tag{2}$$

where $\vec{J}$ is the impressed current source. (2) reduces to (1b) when $\vec{J} = 0$.

B. Basic Equations and Dyadic Green's Functions

The basic equations governing a time harmonic electric and magnetic fields (with the $e^{-j\omega t}$ dependence) inside a chirowaveguide can be obtained by using the constitutive equations in (1a) and (2) in the manipulation of Maxwell's equations. Putting (1a) and (2) into Maxwell's equations, we get

$$\nabla \times \vec{E}(\vec{R}) = \omega\mu\gamma\vec{E}(\vec{R}) + j\omega\mu\vec{H}(\vec{R}) + j\frac{\mu\gamma}{\varepsilon}\vec{J}(\vec{R}) \quad (3a)$$

$$\nabla \times \vec{H}(\vec{R}) = -j\omega\varepsilon\left(1+\frac{\mu\gamma^2}{\varepsilon}\right)\vec{E}(\vec{R}) + \omega\mu\gamma\vec{H}(\vec{R}) + \left(1+\frac{\mu\gamma^2}{\varepsilon}\right)\vec{J}(\vec{R}) \quad (3b)$$

where $\vec{E}$ is the electric field and $\vec{H}$ is the magnetic field. Taking curls of both sides of (3a) and (3b) and with some simple substitutions, we obtain the vector wave equations

$$\nabla \times \nabla \times \vec{E}(\vec{R}) - 2k\sqrt{\frac{\mu}{\varepsilon}}\gamma\nabla \times \vec{E}(\vec{R}) - k^2\vec{E}(\vec{R}) = \left(j\omega\mu + j\frac{\mu\gamma}{\varepsilon}\nabla\times\right)\vec{J}(\vec{R}) \quad (4a)$$

$$\nabla \times \nabla \times \vec{H}(\vec{R}) - 2k\sqrt{\frac{\mu}{\varepsilon}}\gamma\nabla \times \vec{H}(\vec{R}) - k^2\vec{H}(\vec{R}) = \left(1+\frac{\mu\gamma^2}{\varepsilon}\right)\nabla \times \vec{J}(\vec{R}) \quad (4b)$$

where $k = \omega\sqrt{\mu\varepsilon}$. In fact, equivalent forms of (4a) and (4b) have been derived in [15, p. 21] with another form of constitutive equations and with the additional condition of $\vec{K} = 0$. For (4a) and (4b) to have solutions, appropriate boundary conditions must be imposed. The boundary condition for a chirowaveguide with ideally conducting walls requires that the tangential components of the electric field vanishes on the waveguide walls, i.e.,

$$\hat{n} \times \vec{E}(\vec{R}) = 0 \quad (5)$$

where $\hat{n}$ is an outward-pointed unit normal vector defined on the surface of the waveguide.

Due to the linear property of (4a) and (4b), they admit solutions of the following forms:

$$\vec{E}(\vec{R}) = j\omega\mu \iiint_V \bar{\bar{G}}_e(\vec{R}, \vec{R}') \cdot \vec{J}(\vec{R}')dv' \quad (6a)$$

$$\vec{H}(\vec{R}) = \iiint_V \bar{\bar{G}}_m(\vec{R}, \vec{R}') \cdot \vec{J}(\vec{R}')dv' \quad (6b)$$

where $\bar{\bar{G}}_e$ and $\bar{\bar{G}}_m$ are, respectively, the electric and magnetic dyadic Green's functions, and the integrals are carried over the entire volume of the waveguide. Putting (6) into (3), we obtain

$$\nabla \times \bar{\bar{G}}_e(\vec{R} - \vec{R}') = k\sqrt{\frac{\mu}{\varepsilon}}\gamma\,\bar{\bar{G}}_e(\vec{R} - \vec{R}') + \bar{\bar{G}}_m(\vec{R} - \vec{R}') + \frac{1}{k}\sqrt{\frac{\mu}{\varepsilon}}\gamma\,\bar{\bar{I}}\,\delta(\vec{R} - \vec{R}') \quad (7a)$$

$$\nabla\times \bar{\bar{G}}_m(\vec{R} - \vec{R}') = k^2\left(1+\frac{\mu\gamma^2}{\varepsilon}\right)\bar{\bar{G}}_e(\vec{R} - \vec{R}') + k\sqrt{\frac{\mu}{\varepsilon}}\gamma\,\bar{\bar{G}}_m(\vec{R} - \vec{R}') + \left(1+\frac{\mu\gamma^2}{\varepsilon}\right)\bar{\bar{I}}\,\delta(\vec{R} - \vec{R}') \quad (7b)$$

where $\bar{\bar{I}}$ is the unit dyad and $\delta(\vec{R}, \vec{R}')$ is the three-dimensional (3-D) delta function. Taking curl of both sides of (7a) and (7b) leads to the following differential equations of $\bar{\bar{G}}_e$ and $\bar{\bar{G}}_m$

$$\nabla \times \nabla\times \bar{\bar{G}}_e(\vec{R} - \vec{R}') - 2k\sqrt{\frac{\mu}{\varepsilon}}\gamma\nabla\times \bar{\bar{G}}_e(\vec{R} - \vec{R}') - k^2\,\bar{\bar{G}}_e(\vec{R} - \vec{R}') = \left(1+\frac{1}{k}\sqrt{\frac{\mu}{\varepsilon}}\gamma\nabla\times\right)\bar{\bar{I}}\,\delta(\vec{R} - \vec{R}') \quad (8a)$$

$$\nabla \times \nabla\times \bar{\bar{G}}_m(\vec{R} - \vec{R}') - 2k\sqrt{\frac{\mu}{\varepsilon}}\gamma\nabla\times \bar{\bar{G}}_m(\vec{R} - \vec{R}') - k^2\,\bar{\bar{G}}_m(\vec{R} - \vec{R}') = \left(1+\frac{\mu\gamma^2}{\varepsilon}\right)\nabla\times\,\bar{\bar{I}}\,\delta(\vec{R} - \vec{R}') \quad (8b)$$

with the boundary condition

$$\hat{n} \times \bar{\bar{G}}_e(\vec{R}, \vec{R}') = 0. \quad (9)$$

One more useful relation governing $\bar{\bar{G}}_m$ can be derived following the procedure given by Tai [20, ch. 4]. From the boundary condition on the magnetic field, we have

$$\hat{n} \times (\vec{H}^+ - \vec{H}^-) = \left(1+\frac{\mu\gamma^2}{\varepsilon}\right)\vec{J}_s \quad (10)$$

where $\vec{J}_s$ denotes the surface current density on the boundary and $\vec{H}^+$ and $\vec{H}^-$ denote, respectively, the magnetic field inside and outside the boundary. The factor $(1+\frac{\mu\gamma^2}{\varepsilon})$ included is to account for the chirality of the medium. Putting (6b) into (10), we get

$$\hat{n}\times\left[\bar{\bar{G}}_m^+(\vec{R}, \vec{R}') - \bar{\bar{G}}_m^-(\vec{R}, \vec{R}')\right] = \left(1+\frac{\mu\gamma^2}{\varepsilon}\right)\bar{\bar{I}}_s\,\delta(\vec{r} - \vec{r}') \quad (11)$$

where $\bar{\bar{G}}_m^+$ and $\bar{\bar{G}}_m^-$ are, respectively, the magnetic dyadic Green's functions inside and outside the boundary and $\bar{\bar{I}}_s$ is the 2-D unit dyad defined by

$$\bar{\bar{I}}_s = \bar{\bar{I}} - \hat{n}\hat{n} \quad (12)$$

and $\delta(\vec{r} - \vec{r}')$ denotes the 2-D delta function such that

$$\iint \delta(\vec{r} - \vec{r}')ds' = 1. \quad (13)$$

C. Eigenfunction Expansion

To find the eigenfunction expansion of $\bar{\bar{G}}_e$ and $\bar{\bar{G}}_m$, we need to consider solutions to the homogeneous equations of (4a) and (4b), i.e.,

$$\nabla \times \nabla \times \vec{E}(\vec{R}) - 2k\sqrt{\frac{\mu}{\varepsilon}}\gamma\nabla \times \vec{E}(\vec{R}) - k^2\vec{E}(\vec{R}) = 0 \quad (14a)$$

$$\nabla \times \nabla \times \vec{H}(\vec{R}) - 2k\sqrt{\frac{\mu}{\varepsilon}}\gamma\nabla \times \vec{H}(\vec{R}) - k^2\vec{H}(\vec{R}) = 0. \quad (14b)$$

Solutions to (14a) and (14b) in an unbounded chiral medium have been shown to be linear combinations of the $\vec{M}$ type and $\vec{N}$ type vector wave functions [23], [16], [19]. In fact, solutions in a bounded chiral medium such as inside a chirowaveguide are also linear combinations of the $\vec{M}, \vec{N}$ vector wave functions provided that they satisfy additionally the boundary condition in (5). Using Bohren's method [23], we write (3a) and (3b) as

$$\nabla \times \begin{bmatrix} \vec{E}(\vec{R}) \\ \vec{H}(\vec{R}) \end{bmatrix} = \mathbf{K} \begin{bmatrix} \vec{E}(\vec{R}) \\ \vec{H}(\vec{R}) \end{bmatrix} \tag{15a}$$

where

$$\mathbf{K} = \begin{bmatrix} \omega\mu\gamma & j\omega\mu \\ -j\omega\varepsilon\left(1 + \frac{\mu\gamma^2}{\varepsilon}\right) & \omega\mu\gamma \end{bmatrix} \tag{15b}$$

when we put $\vec{J} = 0$ in (3a) and (3b). A linear transformation of the electromagnetic field of the following form:

$$\begin{bmatrix} \vec{E}(\vec{R}) \\ \vec{H}(\vec{R}) \end{bmatrix} = \mathbf{X} \begin{bmatrix} \vec{Q}_1(\vec{R}) \\ \vec{Q}_2(\vec{R}) \end{bmatrix} \tag{16a}$$

where

$$\mathbf{X} = \begin{bmatrix} \frac{j}{\sqrt{\frac{\varepsilon}{\mu} + \gamma^2}} & 1 \\ 1 & j\sqrt{\frac{\varepsilon}{\mu} + \gamma^2} \end{bmatrix} \tag{16b}$$

diagonalizes $\mathbf{K}$. That is, we have

$$\mathbf{X}^{-1}\mathbf{K}\mathbf{X} = \begin{bmatrix} k_+ & 0 \\ 0 & -k_- \end{bmatrix} \tag{17a}$$

where k_+ and k_- are given by

$$k_+ = k\left(\sqrt{\frac{\mu}{\varepsilon}}\gamma + \sqrt{1 + \frac{\mu\gamma^2}{\varepsilon}}\right) \tag{17b}$$

$$k_- = k\left(-\sqrt{\frac{\mu}{\varepsilon}}\gamma + \sqrt{1 + \frac{\mu\gamma^2}{\varepsilon}}\right). \tag{17c}$$

Notice that the choice of $\mathbf{X}$ in (16) is not unique but our results do not depend on a particular choice. $\vec{Q}_1$ and $\vec{Q}_2$ are called the combined fields and when $\mathbf{K}$ is diagonalized as in (17a), they satisfy the following two equations:

$$\nabla \times \begin{bmatrix} \vec{Q}_1(\vec{R}) \\ \vec{Q}_2(\vec{R}) \end{bmatrix} = \begin{bmatrix} k_+ & 0 \\ 0 & -k \end{bmatrix} \begin{bmatrix} \vec{Q}_1(\vec{R}) \\ \vec{Q}_2(\vec{R}) \end{bmatrix} \tag{18}$$

$$\nabla \times \nabla \times \begin{bmatrix} \vec{Q}_1(\vec{R}) \\ \vec{Q}_2(\vec{R}) \end{bmatrix} = \begin{bmatrix} k_+^2 & 0 \\ 0 & k_-^2 \end{bmatrix} \begin{bmatrix} \vec{Q}_1(\vec{R}) \\ \vec{Q}_2(\vec{R}) \end{bmatrix}. \tag{19}$$

From (18) and (19), we see that solutions to the combined fields $\vec{Q}_1$ and $\vec{Q}_2$ are linear combinations of the $\vec{M}, \vec{N}$ vector wave functions provided that they are generated from the same scalar function. So we have

$$\vec{Q}_1(\vec{R}) = a\vec{M}_1(\vec{R}) + b\vec{N}_1(\vec{R}) \tag{20a}$$

$$\vec{Q}_2(\vec{R}) = c\vec{M}_2(\vec{R}) + d\vec{N}_2(\vec{R}) \tag{20b}$$

where

$$\vec{M}_1(\vec{R}) = \nabla \times [\phi_1(\vec{R})\hat{c}] = \frac{1}{k_+}\nabla \times \vec{N}_1(\vec{R}) \tag{21a}$$

$$\vec{N}_1(\vec{R}) = \frac{1}{k_+}\nabla \times \nabla \times [\phi_1(\vec{R})\hat{c}] = \frac{1}{k_+}\nabla \times \vec{M}_1(\vec{R}) \tag{21b}$$

$$\vec{M}_2(\vec{R}) = \nabla \times [\phi_2(\vec{R})\hat{c}] = -\frac{1}{k_-}\nabla \times \vec{N}_2(\vec{R}) \tag{21c}$$

$$\vec{N}_2(\vec{R}) = -\frac{1}{k_-}\nabla \times \nabla \times [\phi_2(\vec{R})\hat{c}] = -\frac{1}{k_-}\nabla \times \vec{M}_2(\vec{R}). \tag{21d}$$

The unit vector $\hat{c}$ in (21a)–(21d) is the piloting vector and the generating functions, ϕ_1 and ϕ_2, for the vector wave functions must satisfy the following scalar Helmholtz equations

$$\nabla^2\phi_1(\vec{R}) + k_+^2\phi_1(\vec{R}) = 0 \tag{22a}$$

$$\nabla^2\phi_2(\vec{R}) + k_-^2\phi_2(\vec{R}) = 0. \tag{22b}$$

In order that $\vec{Q}_1$ and $\vec{Q}_2$ defined in (20a) and (20b) satisfy (18) and (19), we must have

$$a = b \tag{23a}$$

$$c = d. \tag{23b}$$

Therefore we have solutions to (14a) and (14b) as

$$\begin{bmatrix} \vec{E}(\vec{R}) \\ \vec{H}(\vec{R}) \end{bmatrix} = \mathbf{X} \begin{bmatrix} \vec{Q}_1(\vec{R}) \\ \vec{Q}_2(\vec{R}) \end{bmatrix} \tag{24}$$

where $\vec{Q}_1$ and $\vec{Q}_2$ are given by (20a) and (20b), respectively. The coefficients a, c or b, d can be further determined by using the boundary condition of the electric field on the waveguide walls, i.e.,

$$\hat{n} \times \vec{E}(\vec{R}) = \hat{n} \times [t\vec{Q}_1(\vec{R}) + \vec{Q}_2(\vec{R})] = 0 \tag{25a}$$

where

$$t = \frac{j}{\sqrt{\frac{\varepsilon}{\mu} + \gamma^2}}. \tag{25b}$$

Actually the $\vec{Q}_1$ and $\vec{Q}_2$ so obtained in (20) are eigenfunctions to (14). They are also mutually orthogonal as shown later. Thus an arbitrary time harmonic electric or magnetic field inside a chirowaveguide can be expanded by a linear combination of $\vec{Q}_1$ and $\vec{Q}_2$. In view of (8a) and (8b), the dyadic Green's functions $\overline{\overline{G}}_e$ and $\overline{\overline{G}}_m$ can also be expanded by these eigenfunctions. For $\overline{\overline{G}}_e$, the expansion is only valid outside the source point. The reason is that the solenoidal vector wave functions $\vec{M}, \vec{N}$ (thus $\vec{Q}_1, \vec{Q}_2$) are not sufficient because $\overline{\overline{G}}_e$ has a longitudinal part as well and the $\vec{L}$ type wave function is also needed for a complete expansion [18]. However we can expand $\overline{\overline{G}}_m$ completely in terms of $\vec{Q}_1$ and $\vec{Q}_2$ since $\overline{\overline{G}}_m$ is a pure solenoidal dyadic function. When $\overline{\overline{G}}_m$ is known, $\overline{\overline{G}}_e$ can then be obtained from (7b). To obtain the electric dyadic Green's function in this way is termed the method of $\overline{\overline{G}}_m$ as introduced by Tai [20]. The discontinuous nature of the magnetic dyadic Green's function across the source point as shown in (11) is the immediate reason leading to the singular term in the electric dyadic Green's function when the electric dyadic Green's function is derived through the method of $\overline{\overline{G}}_m$. In the following paragraphs, we illustrate the above theoretical formulation by a specific application to a cylindrical chirowaveguide. Because the above formulation has not been restricted to any specific type of chirowaveguides provided that they are filling with a lossless, reciprocal and homogeneous chiral medium and satisfy the boundary condition in (5), it is applicable to all chirowaveguides if they satisfy such a provision.

III. Application to a Cylindrical Chirowaveguide

Consider a cylindrical waveguide of radius a, filled with an isotropic chiral medium and with an ideally conducting wall. Using the cylindrical coordinate system defined for cylindrical waveguides in the usual sense, $\phi_1, \phi_2, \vec{Q}_1$ and $\vec{Q}_2$ are found to be

$$\phi_{1\lambda_1 n}(h) = J_n(\lambda_1 r)e^{jn\phi}e^{jhz} \tag{26a}$$

$$\phi_{2\lambda_2 n}(h) = J_n(\lambda_2 r)e^{jn\phi}e^{jhz} \tag{26b}$$

$$\vec{Q}_{1\lambda_1 n}(h) = A_{\lambda_1 n}\left[\vec{M}_{1\lambda_1 n}(h) + \vec{N}_{1\lambda_1 n}(h)\right] \tag{27a}$$

$$\vec{Q}_{2\lambda_2 n}(h) = B_{\lambda_2 n}\left[\vec{M}_{2\lambda_2 n}(h) + \vec{N}_{2\lambda_2 n}(h)\right] \tag{27b}$$

where

$$\vec{M}_{1\lambda_1 n}(h) = \nabla \times [\phi_{1\lambda_1 n}(h)\hat{z}] = \begin{bmatrix} jn\frac{J_n(\lambda_1 r)}{r} \\ -\frac{\partial J_n(\lambda_1 r)}{\partial r} \\ 0 \end{bmatrix} e^{jn\phi}e^{jhz} = \frac{1}{k_+}\nabla \times \vec{N}_{1\lambda_1 n}(h) \tag{28a}$$

$$\vec{N}_{1\lambda_1 n}(h) = \frac{1}{k_+}\nabla \times \nabla[\phi_{1\lambda_1 n}(h)\hat{z}] = \frac{1}{k_+}\begin{bmatrix} jn\frac{\partial J_n(\lambda_1 r)}{\partial r} \\ -nh\frac{J_n(\lambda_1 r)}{r} \\ \lambda_1^2 J_n(\lambda_1 r) \end{bmatrix} e^{jn\phi}e^{jhz} = \frac{1}{k_+}\nabla \times \vec{M}_{1\lambda_1 n}(h) \tag{28b}$$

$$\vec{M}_{2\lambda_2 n}(h) = \nabla \times [\phi_{2\lambda_2 n}(h)\hat{z}] = \begin{bmatrix} jn\frac{J_n(\lambda_2 r)}{r} \\ -\frac{\partial J_n(\lambda_2 r)}{\partial r} \\ 0 \end{bmatrix} e^{jn\phi}e^{jhz} = -\frac{1}{k_-}\nabla \times \vec{N}_{2\lambda_2 n}(h) \tag{28c}$$

$$\vec{N}_{2\lambda_2 n}(h) = -\frac{1}{k_-}\nabla \times \nabla \times [\phi_{2\lambda_2 n}(h)\hat{z}] = -\frac{1}{k_-}\begin{bmatrix} jh\frac{\partial J_n(\lambda_2 r)}{\partial r} \\ -nh\frac{J_n(\lambda_2 r)}{r} \\ \lambda_2^2 J_n(\lambda_2 r) \end{bmatrix} e^{jn\phi}e^{jhz} = -\frac{1}{k_-}\nabla \times \vec{M}_{2\lambda_2 n}(h). \tag{28d}$$

From (26)–(28), the subscripts λ_1, λ_2, and n attached to the vector functions designating discrete eigenvalues and h is determined from the dispersion equations $\lambda_1^2 + h^2 = k_+^2$, $\lambda_2^2 + h^2 = k_-^2$. $J_n(\lambda_1 r)$ and $J_n(\lambda_2 r)$ are Bessel functions of the first kind and of the order n. The coefficients $A_{\lambda_1 n}$ and $B_{\lambda_2 n}$ and the eigenvalues λ_1 and λ_2 are determined by matching the boundary condition of the electric field on the surface of the cylindrical chirowaveguide. Using (25a), we have

$$\begin{bmatrix} \frac{t\lambda_1^2}{k_+}J_n(\lambda_1 a) & -\frac{\lambda_2^2}{k_-}J_n(\lambda_2 a) \\ t\left[\frac{\partial J_n(\lambda_1 a)}{\partial a} + \frac{nh}{k_+}\frac{J_n(\lambda_1 a)}{a}\right] & \frac{\partial J_n(\lambda_2 a)}{\partial a} - \frac{nh}{k_-}\frac{J_n(\lambda_2 a)}{a} \end{bmatrix} \times \begin{bmatrix} A_{\lambda_1 n} \\ B_{\lambda_2 n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{29}$$

where $\frac{\partial Jn(\lambda_{1,2}a)}{\partial a} = \frac{\partial Jn(\lambda_{1,2}r)}{\partial r}|_{r=a}$. For (29) to have nontrivial solutions for $A_{\lambda_1 n}$ and $B_{\lambda_2 n}$, the coefficient matrix must be singular, which in turn requires

$$\lambda_1^2 J_n(\lambda_1 a)\left[\frac{nh}{a}J_n(\lambda_2 a) - k_-\frac{\partial J_n(\lambda_2 a)}{\partial a}\right] - \lambda_2^2 J_n(\lambda_2 a)\left[\frac{nh}{a}J_n(\lambda_1 a) + k_+\frac{\partial J_n(\lambda_1 a)}{\partial a}\right] = 0. \tag{30}$$

Equation (30) has been derived by P. K. Koivisto *et al.* [9]. When (30) is satisfied, $A_{\lambda_1 n}$ and $B_{\lambda_2 n}$ can be determined as

$$A_{\lambda_1 n} = \begin{cases} \frac{k_+\lambda_2^2 J_n(\lambda_2 a)}{tk_-\lambda_1^2 J_n(\lambda_1 a)}, & \text{when } J_n(\lambda_1 a) \neq 0 \\ -\frac{\frac{\partial J_n(\lambda_2 a)}{\partial a}}{t\frac{\partial J_n(\lambda_1 a)}{\partial a}}, & \text{when } J_n(\lambda_1 a) = 0 \end{cases} \tag{31a}$$

$$B_{\lambda_2 n} = 1. \tag{31b}$$

Hence the modes (or eigenfunctions) of the electromagnetic field inside the waveguide, $\vec{E}_{\lambda_1\lambda_2 n}(\pm h)$ and $\vec{H}_{\lambda_1\lambda_2 n}(\pm h)$, can be represented by

$$\vec{E}_{\lambda_1\lambda_2 n}(\pm h) = t\vec{Q}_{1\lambda_1 n}(\pm h) + \vec{Q}_{2\lambda_2 n}(\pm h), \quad z \gtrless z' \tag{32a}$$

$$\vec{H}_{\lambda_1\lambda_2 n}(\pm h) = \vec{Q}_{1\lambda_1 n}(\pm h) - \tfrac{1}{t}\vec{Q}_{2\lambda_2 n}(\pm h), \quad z \gtrless z'. \tag{32b}$$

The upper lines in the above equations are for modes propagating in the positive z direction while the lower lines are for those propagating in the negative z direction. These modes have been proved to be mutually orthogonal [(21) in [24]] and so a time harmonic electromagnetic field inside the waveguide, $\vec{E}$ and $\vec{H}$, satisfying the Maxwell's equations and the boundary condition in (5) can be expanded as

$$\vec{E}(\vec{R}) = \sum_{\lambda_1\lambda_2 n} \Gamma_{\lambda_1\lambda_2 n}(\pm h)\left[t\vec{Q}_{1\lambda_1 n}(\pm h) + \vec{Q}_{2\lambda_2 n}(\pm h)\right] \quad z \gtrless z' \tag{33a}$$

$$\vec{H}(\vec{R}) = \sum_{\lambda_1\lambda_2 n} \Gamma_{\lambda_1\lambda_2 n}(\pm h)\left[\vec{Q}_{1\lambda_1 n}(\pm h) - \frac{1}{t}\vec{Q}_{2\lambda_2 n}(\pm h)\right] \quad z \gtrless z'. \tag{33b}$$

Note that in (33a) and (33b), n can take both positive and negative integral values for a particular h. The coefficients $\Gamma_{\lambda_1\lambda_2 n}(\pm h)$ can be determined by the method given by R. E. Collin [25]. We consider the expansions

$$\begin{aligned} \nabla \cdot &\left[\vec{E}_{\lambda_1\lambda_2 n}(\pm h) \times \vec{H} - \vec{E} \times \vec{H}_{\lambda_1\lambda_2 n}(\pm h)\right] \\ &= \vec{H} \cdot \nabla \times \vec{E}_{\lambda_1\lambda_2 n}(\pm h) - \vec{E}_{\lambda_1\lambda_2 n}(\pm h) \cdot \nabla \times \vec{H} \\ &\quad - \vec{H}_{\lambda_1\lambda_2 n}(\pm h) \cdot \nabla \times \vec{E} - \vec{E} \cdot \nabla \times \vec{H}_{\lambda_1\lambda_2 n}(\pm h) \\ &= \left[\left(1 + \frac{\mu\gamma^2}{\varepsilon}\right)\vec{E}_{\lambda_1\lambda_2 n}(\pm h) + j\frac{\mu\gamma}{\gamma}\vec{H}_{\lambda_1\lambda_2 n}(\pm h)\right] \cdot \vec{J} \end{aligned} \tag{34}$$

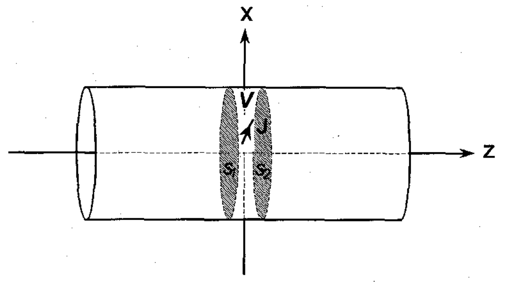

Fig. 1. The volume V containing a current source.

where we have replaced the curl terms of $\vec{E}$ and $\vec{H}$ by their equivalents in (3a) and (3b) and the curl terms of $\vec{E}_{\lambda_1\lambda_2 n}(\pm h)$ and $\vec{H}_{\lambda_1\lambda_2 n}(\pm h)$ by their equivalents in (15a). Integrating both sides of (34) over a volume V that contains the current source $\vec{J}$ and is bounded by the waveguide wall and the two cross-sectional surfaces as shown in Fig. 1, we have

$$\oiint_S [\vec{E}_{\lambda_1\lambda_2 n}(\pm h) \times \vec{H} - \vec{E} \times \vec{H}_{\lambda_1\lambda_2 n}(\pm h)] \cdot \hat{R} ds = \iiint_V \left[\left(1 + \frac{\mu\gamma^2}{\varepsilon}\right)\vec{E}_{\lambda_1\lambda_2 n}(\pm h) + j\frac{\mu\gamma}{\varepsilon}\vec{H}_{\lambda_1\lambda_2 n}(\pm h)\right] \cdot \vec{J} dv \tag{35}$$

where s is the surface enclosing V and $\hat{R}$ is a unit normal vector defined on s and pointed inward to V. In (35), we have used the curl theorem to convert the left-hand side volume integral to a surface integral. Since the integrand of the surface integral vanishes on the waveguide wall, i.e.,

$$\begin{aligned}&[\vec{E}_{\lambda_1\lambda_2 n}(\pm h) \times \vec{H} - \vec{E} \times \vec{H}_{\lambda_1\lambda_2 n}(\pm h)] \cdot \hat{R} \\ &= -[\vec{E}_{\lambda_1\lambda_2 n}(\pm h) \times \hat{n}] \cdot \vec{H} + [\vec{E} \times \hat{n}] \cdot \vec{H}_{\lambda_1\lambda_2 n}(\pm h) \\ &= 0\end{aligned} \tag{36}$$

the only contribution to the surface integral is from the two cross-sectional surfaces (s_1 and s_2 in Fig. 1). To find $\Gamma_{\lambda_1\lambda_2 n}(\pm h)$, we need two orthogonal relations. One is

$$\iint_{S_a} (\vec{e}_p^{\pm} \times \vec{h}_q^{\pm} - \vec{e}_q^{\pm} \times \vec{h}_p^{\pm}) \cdot \hat{z} ds = 0 \tag{37a}$$

from (23) in [24]. The surface integral is carried out over the cross-sectional surface, S_a of the waveguide. The symbols $\vec{e}_p^{\pm}$ and $\vec{e}_q^{\pm}$ denote that part of the electric field and $\vec{h}_p^{\pm}$and$\vec{h}_q^{\pm}$ that part of the magnetic field that depend on the transverse coordinates (r, ϕ) only. The superscript + (or −) sign means that the fields are propagating in the positive (or negative) z direction. The subscripts p and q represent different eigenmodes and the asterisk denotes complex conjugation. The other orthogonal realation is

$$\iint_{S_a} (\vec{e}_p^{\pm} \times \vec{h}_q^{\pm} - \vec{e}_q^{\pm} \times \vec{h}_p^{\pm}) \cdot \hat{z} ds = 0, \qquad \text{when } p \neq q \tag{37b}$$

which can be similarly derived as (37a). Substitute the expressions of $\vec{E}$ and $\vec{H}$ in (33a) and (33b) into the left-hand side of (35) and consider the $\vec{E}_{\lambda_1\lambda_2(-n)}(h)$ and $\vec{H}_{\lambda_1\lambda_2(-n)}(h)$ modes. We then have by using the orthogonal relations in (37a) and (37b)

$$\begin{aligned}&\Gamma_{\lambda_1\lambda_2 n}(-h) \iint_{S_1} [\vec{E}_{\lambda_1\lambda_2(-n)}(h) \times \vec{H}_{\lambda_1\lambda_2 n}(-h) \\ &\qquad - \vec{E}_{\lambda_1\lambda_2 n}(-h) \times \vec{H}_{\lambda_1\lambda_2(-n)}(h)] \cdot \hat{z} ds \\ &= \iiint_V \left[\left(1 + \frac{\mu\gamma^2}{\varepsilon}\right)\vec{E}_{\lambda_1\lambda_2(-n)}(h) + j\frac{\mu\gamma}{\varepsilon}\vec{H}_{\lambda_1\lambda_2(-n)}(h)\right] \cdot \vec{J} dv\end{aligned} \tag{38}$$

where the surface integral over the cross-sectional surface s_2 has been evaluated to zero because

$$\iint_{S_2} \left[\vec{E}_{\lambda_1\lambda_2(-n)}(h) \times \sum_{\lambda_1\lambda_2 n} \Gamma_{\lambda_1\lambda_2 n}(h)\vec{H}_{\lambda_1\lambda_2 n}(h) - \sum_{\lambda_1\lambda_2 n} \Gamma_{\lambda_1\lambda_2 n}(h)\vec{E}_{\lambda_1\lambda_2 n}(h) \times \vec{H}_{\lambda_1\lambda_2(-n)}(h)\right] \cdot (-\hat{z}) ds = 0 \tag{39}$$

by using another orthogonal relation in (23) of [24]. Denote the surface integral on the left-hand side of (38) by $I_{\lambda_1\lambda_2 n}(-h)$, which can be evaluated to be

$$\begin{aligned}&I_{\lambda_1\lambda_2 n}(-h) \\ &= \iint_{S_1} [\vec{E}_{\lambda_1\lambda_2(-n)}(h) \times \vec{H}_{\lambda_1\lambda_2 n}(-h) \\ &\qquad - \vec{E}_{\lambda_1\lambda_2 n}(-h) \times \vec{H}_{\lambda_1\lambda_2(-n)}(h)] \cdot \hat{z} ds \\ &= (-1)^n 8\pi j \left\{ tA_{\lambda_1 n}^2 \left\{ \left(1 + \frac{h^2}{k_+^2}\right) \frac{n}{2} [J_n^2(\lambda_1 a) - \delta_{n0}] \right.\right. \\ &\qquad - \frac{a^2 h}{4k_+} \left\{ \left[\lambda_1 - \frac{(n-1)^2}{a^2}\right] J_{n-1}^2(\lambda_1 a) \right. \\ &\qquad + \left[\frac{\partial J_{n-1}(\lambda_1 a)}{\partial a}\right]^2 + \left[\lambda_1 - \frac{(n+a)^2}{a^2}\right] \\ &\qquad \left.\left. \times J_{n+1}^2(\lambda_1 a) + \left[\frac{\partial J_{n+1}(\lambda_1 a)}{\partial a}\right]^2 \right\}\right\} \\ &\qquad - \frac{B_{\lambda_1 n}^2}{t} \left\{ \left(1 + \frac{h^2}{k_-^2}\right) \frac{n}{2} [J_n^2(\lambda_2 a) - \delta_{n0}] \right. \\ &\qquad + \frac{a^2 h}{4k_-} \left\{ \left[\lambda_2 - \frac{(n-1)^2}{a^2}\right] \right. \\ &\qquad \times J_{n-1}^2(\lambda_2 a) + \left[\frac{\partial J_{n-1}(\lambda_2 a)}{\partial a}\right]^2 \\ &\qquad + \left[\lambda_2 - \frac{(n+1)^2}{a^2}\right] J_{n+1}^2(\lambda_2 a) \\ &\qquad \left.\left.\left. + \left[\frac{\partial J_{n+1}(\lambda_2 a)}{\partial a}\right]^2 \right\}\right\}\right\}\end{aligned} \tag{40}$$

where δ_{n0} is the Kronecker delta defined with respect to n, i.e., $\delta_{n0} = \begin{cases} 1, & \text{when } n = 0 \\ 0, & \text{when } n \neq 0 \end{cases}$.

Hence

$$\Gamma_{\lambda_1\lambda_2 n}(-h) = \frac{\iiint_V \left[\left(1+\frac{\mu\gamma^2}{\varepsilon}\right)\vec{E}_{\lambda_1\lambda_2(-n)}(h) + j\frac{\mu\gamma}{\varepsilon}\vec{H}_{\lambda_1\lambda_2(-n)}(h)\right]\cdot\vec{J}dv}{I_{\lambda_1\lambda_2 n}(-h)}. \tag{41}$$

By a similar reasoning but considering the $\vec{E}_{\lambda_1\lambda_2(-n)}(-h)$ and $\vec{H}_{\lambda_1\lambda_2(-n)}(-h)$ modes instead, we have

$$\begin{aligned} I_{\lambda_1\lambda_2 n}(h) &= \iint_{S_2} [\vec{E}_{\lambda_1\lambda_2(-n)}(-h)\times\vec{H}_{\lambda_1\lambda_2 n}(h) \\ &\quad - \vec{E}_{\lambda_1\lambda_2 n}(h)\times\vec{H}_{\lambda_1\lambda_2(-n)}(-h)]\cdot(-\hat{z})ds \\ &= -(-1)^n 8\pi j\Bigg\{tA^2\Bigg\{\left(1+\frac{h^2}{k_+^2}\right)\frac{n}{2}[J_n^2(\lambda_1 a)-\delta_{n0}] \\ &\quad + \frac{a^2h}{4k_+}\Bigg\{\left[\lambda_1 - \frac{(n-1)^2}{a^2}\right]J_{n-1}^2(\lambda_1 a) \\ &\quad + \left[\frac{\partial J_{n-1}(\lambda_1 a)}{\partial a}\right]^2 + \left[\lambda_1 - \frac{(n+1)^2}{a^2}\right] \\ &\quad \times J_{n+1}^2(\lambda_1 a) + \left[\frac{\partial J_{n+1}(\lambda_1 a)}{\partial a}\right]^2\Bigg\}\Bigg\} \\ &\quad - \frac{B^2}{t}\Bigg\{\left(1+\frac{h^2}{k_-^2}\right)\frac{n}{2}[J_n^2(\lambda_2 a)-\delta_{n0}] \\ &\quad - \frac{a^2h}{4k_-}\Bigg\{\left[\lambda_2 - \frac{(n-1)^2}{a^2}\right] \\ &\quad \times J_{n-1}^2(\lambda_2 a) + \left[\frac{\partial J_{n-1}(\lambda_2 a)}{\partial a}\right]^2 \\ &\quad + \left[\lambda_2 - \frac{(n+1)^2}{a^2}\right]J_{n+1}^2(\lambda_2 a) \\ &\quad + \left[\frac{\partial J_{n+1}(\lambda_2 a)}{\partial a}\right]^2\Bigg\}\Bigg\}\Bigg\}. \end{aligned} \tag{42}$$

(The evaluations of (40) and (42) are given in the Appendix). So

$$\Gamma_{\lambda_1\lambda_2 n}(h) = \frac{\iiint_V \left[\left(1+\frac{\mu\gamma^2}{\varepsilon}\right)\vec{E}_{\lambda_1\lambda_2(-n)}(-h) + j\frac{\mu\gamma}{\varepsilon}\vec{H}_{\lambda_1\lambda_2(-n)}(-h)\right]\cdot\vec{J}dv}{I_{\lambda_1\lambda_2 n}(h)}. \tag{43}$$

Now we can go to find $\overline{\overline{G}}_m$ which can be obtained from the expansion of $\vec{H}$ in (33b). The magnetic dyadic Green's function $\overline{\overline{G}}_m$ is defined in (6b). By equating the right-hand sides of (6b) and (33b), we have as shown in (44) at the bottom of the page, where we have used the primed functions to designate that they are defined with respect to the source coordinates. From (44) at the bottom of the page, we immediately see as that in (45) shown at the bottom of the next page.

Although the expansion of the magnetic field in (33b) is defined only outside the source point, its singularity in the source point is of the order $1/r^2$ (the $1/r^2$ factor resulting from the product terms of $\vec{M}\vec{M}'$, $\vec{M}\vec{N}'$, $\vec{N}\vec{M}'$ or $\vec{N}\vec{N}'$ in (33b) provided that the current source dose not introduce another singularity at the source point). Hence $\overline{\overline{G}}_m$, which has the same order of singularity as the magnetic field, is still integrable even at the source point [26].

In using the method of $\overline{\overline{G}}_m$ to find the electric dyadic Green's function, the key step is to obtain an expression of $\nabla\times\overline{\overline{G}}_m$ while taking into consideration of the discontinuous behavior of $\overline{\overline{G}}_m$ at $z = z'$. Following exactly the same steps as in [20, ch. 5] but using the discontinuous property of $\overline{\overline{G}}_m$ in (11) instead, we have

$$\begin{aligned} \nabla\times\overline{\overline{G}}_m(\vec{R}-\vec{R}') &= [\nabla\times\overline{\overline{G}}{}_m^{+}(\vec{R}-\vec{R}')]U(z-z') \end{aligned}$$

$$\begin{aligned} \iiint_V \overline{\overline{G}}_m(\vec{R},\vec{R}')\cdot\vec{J}(\vec{R}')dv' &= \sum_{\lambda_1\lambda_2 n}\Gamma_{\lambda_1\lambda_2 n}(\pm h)\left[\vec{Q}_{1\lambda_1 n}(\pm h) - \frac{1}{t}\vec{Q}_{2\lambda_2 n}(\pm h)\right] \qquad z \gtrless z' \\ &= \sum_{\lambda_1\lambda_2 n}\left[\vec{Q}_{1\lambda_1 n}(\pm h) - \frac{1}{t}\vec{Q}_{2\lambda_2 n}(\pm h)\right] \\ &\quad \times \frac{\iiint_V \left[\left(1+\frac{\mu\gamma^2}{\varepsilon}\right)\vec{E}'_{\lambda_1\lambda_2(-n)}(\mp h) + j\frac{\mu\gamma}{\varepsilon}\vec{H}'_{\lambda_1\lambda_2(-n)}(\mp h)\right]\cdot\vec{J}'dv'}{I_{\lambda_1\lambda_2 n}(\mp h)} \qquad z \gtrless z' \\ &= \sum_{\lambda_1\lambda_2 n}\left[\vec{Q}_{1\lambda_1 n}(\pm h) - \frac{1}{t}\vec{Q}_{2\lambda_2 n}(\pm h)\right] \\ &= \frac{\iiint_V \left\{\left[t\left(1+\frac{\mu\gamma^2}{\varepsilon}\right) + j\frac{\mu\gamma}{\varepsilon}\right]\vec{Q}'_{1\lambda_1(-n)}(\mp h) + \left[\left(1+\frac{\mu\gamma^2}{\varepsilon}\right) - j\frac{\mu\gamma}{t\varepsilon}\right]\vec{Q}'_{2\lambda_2(-n)}(\mp h)\right\}\cdot\vec{J}'dv'}{I_{\lambda_1\lambda_2 n}(\mp h)} \qquad z \gtrless z' \end{aligned} \tag{44}$$

$$+ \left[\nabla \times \bar{\bar{G}}_m^-(\vec{R}-\vec{R}')\right]U(z'-z) + \left(1+\frac{\mu\gamma^2}{\varepsilon}\right)(\bar{\bar{I}} - \hat{z}\hat{z})\delta(\vec{R}-\vec{R}') \tag{46}$$

where $\bar{\bar{G}}_m^+$ and $\bar{\bar{G}}_m^-$ are now the magnetic dyadic Green's functions for $z > z'$ and $z < z'$, respectively, and U is the unit step function. Substituting (46) and (45) into (7b), we obtain

$$\begin{aligned}
&\bar{\bar{G}}_e(\vec{R}-\vec{R}') \\
&= \frac{1}{k^2\left(1+\frac{\mu\gamma^2}{\varepsilon}\right)}\nabla\times \bar{\bar{G}}_m(\vec{R}-\vec{R}') \\
&\quad - \frac{\sqrt{\frac{\mu}{\varepsilon}}\gamma}{k\left(1+\frac{\mu\gamma^2}{\varepsilon}\right)}\bar{\bar{G}}_m(\vec{R}-\vec{R}') - \frac{1}{k^2}\bar{\bar{I}}\delta(\vec{R}-\vec{R}') \\
&= -\frac{1}{k^2}\hat{z}\hat{z}\delta(\vec{R}-\vec{R}')\sum_{\lambda_1\lambda_2 n}\frac{1}{I_{\lambda_1\lambda_2 n}(\mp h)k\sqrt{1+\frac{\mu\gamma^2}{\varepsilon}}} \\
&\quad\times\left\{t\left[\left(1+\frac{\mu\gamma^2}{\varepsilon}\right)+j\frac{\mu\gamma}{t\varepsilon}\right]\vec{Q}_{1\lambda_1 n}(\pm h)\vec{Q}'_{1\lambda_1(-n)}(\mp h)\right. \\
&\quad + \left[\left(1+\frac{\mu\gamma^2}{\varepsilon}\right)-j\frac{\mu\gamma}{t\varepsilon}\right]\vec{Q}_{1\lambda_1 n}(\pm h)\vec{Q}'_{2\lambda_2(-n)}(\mp h) \\
&\quad + \left[\left(1+\frac{\mu\gamma^2}{\varepsilon}\right)+j\frac{\mu\gamma}{t\varepsilon}\right]\vec{Q}_{2\lambda_2 n}(\pm h)\vec{Q}'_{1\lambda_1(-n)}(\mp h) \\
&\quad \left.+\frac{1}{t}\left[\left(1+\frac{\mu\gamma^2}{\varepsilon}\right)-j\frac{\mu\gamma}{t\varepsilon}\right]\vec{Q}_{2\lambda_2 n}(\pm h)\vec{Q}'_{2\lambda_2(-n)}(\mp h)\right\} \\
&\qquad z \gtrless z'
\end{aligned} \tag{47}$$

after some simple manipulations. Note that the singular term of $\bar{\bar{G}}_e$ is the same as those appearing in the electric dyadic Green's functions for achiral waveguides [20].

IV. Conclusion

We have laid down a general method of formulating dyadic Green's functions by eigenfunction expansions in homogeneous chirowaveguides. The electric and magnetic dyadic Green's functions for a cylindrical chirowaveguide have been rigorously derived. With these dyadic Green's functions, the problem of radiation by a current source inside a chirowaveguide can be solved. This will help determine the excitation method and feed point impedance of the chirowaveguide.

Appendix

The evaluations of the integrals in (40) and (42), i.e.,

$$\begin{aligned}
&I_{\lambda_1\lambda_2 n}(\mp h) \\
&= \iint_{S_1,S_2}\left[\vec{E}_{\lambda_1\lambda_2(-n)}(\pm h)\times\vec{H}_{\lambda_1\lambda_2 n}(\mp h)\right. \\
&\qquad \left.-\vec{E}_{\lambda_1\lambda_2 n}(\mp h)\times\vec{H}_{\lambda_1\lambda_2(-n)}(\pm h)\right]\cdot(\pm\hat{z})ds \\
&= \int_0^{2\pi}\int_0^a\left[\vec{E}_{\lambda_1\lambda_2(-n)}(\pm h)\times\vec{H}_{\lambda_1\lambda_2 n}(\mp h)\right. \\
&\qquad \left.-\vec{E}_{\lambda_1\lambda_2 n}(\mp h)\times\vec{H}_{\lambda_1\lambda_2(-n)}(\pm h)\right]\cdot(\pm\hat{z})r\,dr\,d\phi
\end{aligned} \tag{48}$$

are given in this Appendix. From (32a) and (32b), we have

$$\begin{aligned}
&\vec{E}_{\lambda_1\lambda_2(-n)}(\pm h)\times\vec{H}_{\lambda_1\lambda_2 n}(\mp h) \\
&= \left[t\vec{Q}_{1\lambda_1(-n)}(\pm h)+\vec{Q}_{2\lambda_2(-n)}(\pm h)\right] \\
&\quad\times\left[\vec{Q}_{1\lambda_1 n}(\mp h)-\frac{1}{t}\vec{Q}_{2\lambda_2 n}(\mp h)\right] \\
&= t\vec{Q}_{1\lambda_1(-n)}(\pm h)\times\vec{Q}_{1\lambda_1 n}(\mp h) \\
&\quad - \vec{Q}_{1\lambda_1(-n)}(\pm h)\times\vec{Q}_{2\lambda_2 n}(\mp h) \\
&\quad + \vec{Q}_{2\lambda_2(-n)}(\pm h)\times\vec{Q}_{1\lambda_1 n}(\mp h) \\
&\quad - \frac{1}{t}\vec{Q}_{2\lambda_2(-n)}(\pm h)\times\vec{Q}_{2\lambda_2 n}(\mp h)
\end{aligned} \tag{49a}$$

$$\begin{aligned}
\bar{\bar{G}}_m(\vec{R},\vec{R}') &= \sum_{\lambda_1\lambda_2 n}\left[\vec{Q}_{1\lambda_1 n}(\pm h)-\frac{1}{t}\vec{Q}_{2\lambda_2 n}(\pm h)\right] \\
&\quad\times\frac{\left\{t\left[\left(1+\frac{\mu\gamma^2}{\varepsilon}\right)+j\frac{\mu\gamma}{t\varepsilon}\right]\vec{Q}'_{1\lambda_1(-n)}(\mp h)+\left[\left(1+\frac{\mu\gamma^2}{\varepsilon}\right)-j\frac{\mu\gamma}{t\varepsilon}\right]\vec{Q}'_{2\lambda_2(-n)}(\mp h)\right\}}{I_{\lambda_1\lambda_2 n}(\mp h)} \qquad z \gtrless z' \\
&= \sum_{\lambda_1\lambda_2 n}\frac{1}{I_{\lambda_1\lambda_2 n}(\mp h)} \\
&\quad\times\left\{t\left[\left(1+\frac{\mu\gamma^2}{\varepsilon}\right)+j\frac{\mu\gamma}{t\varepsilon}\right]\vec{Q}_{1\lambda_1(n)}(\pm h)\vec{Q}'_{1\lambda_1(-n)}(\mp h)\right. \\
&\quad + \left[\left(1+\frac{\mu\gamma^2}{\varepsilon}\right)-j\frac{\mu\gamma}{t\varepsilon}\right]\vec{Q}_{1\lambda_1(n)}(\pm h)\vec{Q}'_{2\lambda_2(-n)}(\mp h) \\
&\quad - \left[\left(1+\frac{\mu\gamma^2}{\varepsilon}\right)+j\frac{\mu\gamma}{t\varepsilon}\right]\vec{Q}_{2\lambda_2(n)}(\pm h)\vec{Q}'_{1\lambda_1(-n)}(\mp h) \\
&\quad \left.-\frac{1}{t}\left[\left(1+\frac{\mu\gamma^2}{\varepsilon}\right)-j\frac{\mu\gamma}{t\varepsilon}\right]\vec{Q}_{2\lambda_2(n)}(\pm h)\vec{Q}'_{2\lambda_2(-n)}(\mp h)\right\} \\
&\qquad z \gtrless z'
\end{aligned} \tag{45}$$

$$\begin{aligned}&\vec{E}_{\lambda_1\lambda_2 n}(\mp h)\times\vec{H}_{\lambda_1\lambda_2(-n)}(\pm h)\\&=\left[t\vec{Q}_{1\lambda_1 n}(\mp h)+\vec{Q}_{2\lambda_2 n}(\mp h)\right]\\&\quad\times\left[\vec{Q}_{1\lambda_1(-n)}(\pm h)-\frac{1}{t}\vec{Q}_{2\lambda_2(-n)}(\pm h)\right]\\&=t\vec{Q}_{1\lambda_1 n}(\mp h)\times\vec{Q}_{1\lambda_1(-n)}(\pm h)\\&\quad-\vec{Q}_{1\lambda_1 n}(\mp h)\times\vec{Q}_{2\lambda_2(-n)}(\pm h)\\&\quad+\vec{Q}_{2\lambda_2 n}(\mp h)\times\vec{Q}_{1\lambda_1(-n)}(\pm h)\\&\quad-\frac{1}{t}\vec{Q}_{2\lambda_2 n}(\mp h)\times\vec{Q}_{2\lambda_2(-n)}(\pm h).\end{aligned}\tag{49b}$$

Hence

$$\begin{aligned}&\left[\vec{E}_{\lambda_1\lambda_2(-n)}(\pm h)\times\vec{H}_{\lambda_1\lambda_2 n}(\mp h)\right.\\&\quad\left.-\vec{E}_{\lambda_1\lambda_2 n}(\mp h)\times\vec{H}_{\lambda_1\lambda_2(-n)}(\pm h)\right]\cdot(\pm\hat{z})\\&=2\left[t\vec{Q}_{1\lambda_1(-n)}(\pm h)\times\vec{Q}_{1\lambda_1 n}(\mp h)\right.\\&\quad\left.-\frac{1}{t}\vec{Q}_{2\lambda_2(-n)}(\pm h)\times\vec{Q}_{2\lambda_2 n}(\mp h)\right]\cdot(\pm\hat{z}).\end{aligned}\tag{50}$$

From (27a) and (27b)

$$\begin{aligned}&\vec{Q}_{1\lambda_1(-n)}(\pm h)\times\vec{Q}_{1\lambda_1(n)}(\mp h)\cdot(\pm\hat{z})\\&=A^2_{\lambda_1 n}\left[\vec{M}_{1\lambda_1(-n)}(\pm h)+\vec{N}_{1\lambda(-n)}(\pm h)\right]\\&\quad\times\left[\vec{M}_{1\lambda_1 n}(\mp h)+\vec{N}_{1\lambda_1 n}(\mp h)\right]\cdot(\pm\hat{z})\\&=A^2_{\lambda_1 n}\left[\vec{M}_{1\lambda_1(-n)}(\pm h)\times\vec{M}_{1\lambda_1 n}(\mp h)\right.\\&\qquad+\vec{M}_{1\lambda_1(-n)}(\pm h)\times\vec{N}_{1\lambda_1 n}(\mp h)\\&\qquad+\vec{N}_{1\lambda_1(-n)}(\pm h)\times\vec{M}_{1\lambda_1 n}(\mp h)\\&\qquad\left.+\vec{N}_{1\lambda_1(-n)}(\pm h)\times\vec{N}_{1\lambda_1 n}(\mp h)\right]\cdot(\pm\hat{z})\\&=\pm(-1)^n j2A^2_{\lambda_1 n}\\&\quad\times\left\{\left(1+\frac{h^2}{k_+^2}\right)n\frac{J_n(\lambda_1 r)}{r}\frac{\partial J_n(\lambda_1 r)}{\partial r}\mp\frac{h}{k_+}\right.\\&\qquad\left.\times\left[n^2\left(\frac{J_n(\lambda_1 r)}{r}\right)^2+\left(\frac{\partial J_n(\lambda_1 r)}{\partial r}\right)^2\right]\right\}\end{aligned}\tag{51a}$$

$$\begin{aligned}&\vec{Q}_{2\lambda_2(-n)}(\pm h)\times\vec{Q}_{2\lambda_2 n}(\mp h)\cdot(\pm\hat{z})\\&=B^2_{\lambda_1 n}\left[\vec{M}_{2\lambda_2(-n)}(\pm h)+\vec{N}_{2\lambda_2(-n)}(\pm h)\right]\\&\quad\times\left[\vec{M}_{2\lambda_2 n}(\mp h)+\vec{N}_{2\lambda_2 n}(\mp h)\right]\cdot(\pm\hat{z})\\&=B^2_{\lambda_1 n}\left[\vec{M}_{2\lambda_2(-n)}(\pm h)\times\vec{M}_{2\lambda_2 n}(\mp h)\right.\\&\qquad+\vec{M}_{2\lambda_2(-n)}(\pm h)\times\vec{N}_{2\lambda_2 n}(\mp h)\\&\qquad+\vec{N}_{2\lambda_2(-n)}(\pm h)\times\vec{M}_{2\lambda_2 n}(\mp h)\\&\qquad\left.+\vec{N}_{2\lambda_2(-n)}(\pm h)\times\vec{N}_{2\lambda_2 n}(\mp h)\right]\cdot(\pm\hat{z})\\&=\pm(-1)^n j2B^2_{\lambda_1 n}\\&\quad\times\left\{\left(1+\frac{h^2}{k_-^2}\right)n\frac{J_n(\lambda_2 r)}{r}\frac{\partial J_n(\lambda_2 r)}{\partial r}\pm\frac{h}{k_-}\right.\\&\qquad\left.\times\left[n^2\left(\frac{J_n(\lambda_2 r)}{r}\right)^2+\left(\frac{\partial J(\lambda_2 r)}{\partial r}\right)^2\right]\right\}.\end{aligned}\tag{51b}$$

Putting (51a) and (51b) into (48) and using the following formula [20, p. 137]

$$\int_0^a J_n^2(\alpha r)rdr=\frac{a^2}{2\alpha^2}\left\{\left(\alpha^2-\frac{n^2}{a^2}\right)J_n^2(\alpha a)+\left[\frac{\partial J_n(\alpha a)}{\partial a}\right]^2\right\}\tag{52}$$

we can obtain (40) and (42).

REFERENCES

[1] N. Engheta and P. Pelet, "Modes in chirowaveguides," *Opt. Lett.*, vol. 14, no. 11, pp. 593–595, June 1989.
[2] C. Eftimiu and L. W. Pearson, "Guided electromagnetic waves in chiral media," *Radio Sci.*, vol. 24, no. 3, pp. 351–359, May–June 1989.
[3] P. Pelet and N. Engheta, "The theory of chirowaveguides," *IEEE Trans. Antennas Propagat.*, vol. 38, no. 1, pp. 90–98, Jan. 1990.
[4] ——, "Coupled-mode theory for chirowaveguides," *J. Appl. Phys.*, vol. 67, no. 6, pp. 2742–2745, Mar. 1990.
[5] J. A. M. Svedin, "Propagation analysis of chirowaveguides using the finite-element method," *IEEE Trans. Microwave Theory Tech.*, vol. 38, no. 10, pp. 1488–1496, Oct. 1990.
[6] H. Cory and I. Rosenhouse, "Electromagnetic wave propagation along a chiral slab," *IEE Proc.*, pt. H, Feb. 1991, vol. 138, no. 1, pp. 51–54.
[7] M. Oksanen and P. K. Koivisto, "Dispersion curves and fields for a chiral slab waveguide," *IEE Proc.*, pt. H, Aug. 1991, vol. 138, no. 4, pp. 327–334.
[8] P. Pelet and N. Engheta, "Modal analysis for rectangular chirowaveguides with metallic walls using the finite-difference method," *J. Electromagnetic Waves Applicat.*, vol. 6, no. 9, pp. 1277–1285, 1992.
[9] P. K. Koivisto, S. A. Tretyakov, and M. I. Oksanen, "Waveguides filled with general biisotropic media," *Radio Sci.*, vol. 28, no. 5, pp. 675–686, Sept. 1993.
[10] H. Cory, "Wave propagation along a closed rectangular chirowaveguide," *Microwave Optical Technology Lett.*, vol. 6, no. 14, pp. 797–800, Nov. 1993.
[11] L. Y. Zhang, Y. C. Jiao, and C. H. Liang, "The dominant mode in a parallel-plate chirowaveguide," *IEEE Trans. Microwave Theory Tech.*, vol. 42, no. 10, pp. 2009–2012, Oct. 1994.
[12] H. N. Kritikos and D. L. Jaggard, *Recent Advances in Electromagnetic Theory.* New York: Springer-Verlag, 1990, ch. 1.
[13] I. V. Lindell, A. H. Sihvola, S. A. Tretyakov, and A. J. Viitanen, *Electromagnetic Waves in Chiral and Bi-Isotropic Media.* London: Artech House, 1994, ch. 1.
[14] S. Bassiri, N. Engheta, and C. H. Papas, "Dyadic Green's function and dipole radiation in chiral media," *Alta Freq.*, vol. 55, pp. 83–88, 1986.
[15] A. Lakhtakia, V. K. Varadan, and V. V. Varadan, *Time-Harmonic Electromagnetic Fields in Chiral Media.* Berlin: Springer-Verlag, 1989.
[16] N. Engheta and M. W. Kowarz, "Antenna radiation in the presence of a chiral sphere," *J. Appl. Phys.*, vol. 67, no. 2, pp. 639–647, Jan. 1990.
[17] N. Engheta and S. Bassiri, "One- and two-dimensional dyadic Green's functions in chiral media," *IEEE Trans. Antennas Propagat.*, vol. 37, no. 4, pp. 512–515, Apr. 1989.
[18] C. T. Tai, "On the eigen-function expansion of dyadic Green's functions," *Proc. IEEE*, 1973, vol. 61, pp. 480–481.
[19] L. W. Li, P. S. Kooi, M. S. Leong, and T. S. Yeo, "A general expression of dyadic Green's functions in radially multilayered chiral media," *IEEE Trans. Antennas Propagat.*, vol. 43, no. 3, pp. 232–238, Mar. 1995.
[20] C. T. Tai, *Dyadic Green's Functions in Electromagnetic Theory.* New York: IEEE Press, 1993.
[21] E. J. Post, *Formal Structure of Electromagnetics.* Amsterdam: North-Holland, 1962.
[22] D. J. Jaggard, A. R. Mickelson, and C. H. Papas, "On electromagnetic waves in chiral media," *Appl. Phys.*, vol. 18, pp. 211–216, 1979.
[23] C. F. Bohren, "Light scattering by an optically active sphere," *Chemical Physics Lett.*, vol. 29, no. 3, pp. 458–462, Dec. 1974.
[24] N. Engheta and P. Pelet, "Mode orthogonality in chirowaveguides," *IEEE Trans. Microwave Theory Tech.*, vol. 38, no. 11, pp. 1631–1634, Nov. 1990.
[25] R. E. Collin, *Field Theory of Guided Waves.* New York: IEEE Press, 1991, ch. 5.
[26] O. D. Kellogg, *Foundations of Potential Theory.* New York: Dover, 1953, ch. VI.

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